Assignment 9

Hand in no. 2, 5, and 7 by Nov 21.

1. Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous.

2. Prove that $\{\cos nx\}_{n=1}^{\infty}$ does not have any convergent subsequence in $C[0,1]$.

3. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there are $x_0 \in E$ and a constant $M > 0$ such that $|f(x_0)| \le M$ for all f in this family.

4. Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be

$$F_n(x) = \int_0^x f_n(t)dt.$$

   (a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \le M$ for all n.

   (b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that

$$\int_0^1 |f_n|^2 \le K, \quad \forall n.$$

5. Prove that the set consisting of all functions G of the form

$$G(x) = \sin x + \int_0^x \frac{g(y)}{1+g^2(y)}\, dy\ ,$$

   where $g \in C[0,1]$ is precompact in $C[0,1]$.

6. Let $K \in C([a,b]\times[a,b])$ and define the operator T by

$$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$

   (a) Show that T maps $C[a,b]$ to itself.

   (b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence in the sup-norm.

7. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x-a)$. Show that there exists a sequence of unit intervals $I_k = [n_k, n_k+1]$, $n_k \to \infty$, such that $\{f_{n_k}\}$ converges uniformly on $[0,1]$.

8. A bump function is a smooth function φ in $\mathbb{R}^2$ which is positive in the unit disk, vanishing outside the ball, and satisfies $\iint_{\mathbb{R}^2} \varphi(x)\, dA(x) = 1$. Let f be a continuous function defined in an open set containing $\overline{G}$ where G is bounded and open in $\mathbb{R}^2$. For small $\varepsilon > 0$, define

$$f_\varepsilon(x) = \frac{1}{\varepsilon^2}\iint_{\mathbb{R}^2} \varphi\left(\frac{y-x}{\varepsilon}\right) f(y)\, dA(y)\ .$$

   Show that f_ε is $C^\infty(\overline{G})$ and tends to f uniformly as $\varepsilon \to 0$.